Numerical Simulations of $d = 3$ $SU(2)$ LGT in the Dual Formulation

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We have developed the techniques necessary for a numerical simulation of $d = 3$ $SU(2)$ Lattice Gauge theories in the dual formulation as originally developed by Anishetty et al [1]. These include updating techniques that preserve the constrained configuration space, efficient evaluation of 6-j symbols and a certain problem associated with the positive indefiniteness of the weight factors.

1. Motivation

The outstanding problems in QCD are understanding the mechanisms of confinement, chiral symmetry breaking and the spectrum of the theory including the vacuum structure. Lattice gauge theories have gone a long way in shedding light on some of these important issues. Nevertheless, we are still not in a position to, say, discriminate between different mechanisms for confinement like the dual superconductor mechanism, the Z_N-fluxon mechanisms etc. In particular a detailed picture of QCD in terms of the dual variables is still at a preliminary level. Therefore it is desirable to probe alternative formulations of LGT that place greater emphasis on the dual variables.

The dual variables simplify the description of some observables while complicating some others. In the formulation to be discussed here, we shall see that while the t'Hooft disordered variables become diagonal, the Wilson loop operators become quite complicated. The continuum limit of LGT's is in the extreme weak coupling regime and here many of the gauge invariant observables like plaquettes approach unity so in extracting the scaling part of various correlation lengths large "perturbative" corrections need to be subtracted. But in the dual formulation there are many gauge invariant observables that actually vanish in the weak coupling regime so it can be hoped that correlation functions of these observables are lot "cleaner". Other important feature of the dual formulation is that it provides manifestly gauge invariant characterisation of fluctuations.

As shown by [1] there is an intriguing connection between the dual formulation of Yang-Mills theories in $d = 3$ and the Regge-Ponzano formulation of gravity in $d = 3$. A count of the relevant degrees of freedom indicates that this should happen in $d = 4$ also.

2. The Dual Formulation

In the conventional formulation, group elements are assigned to the links of the lattice and gauge invariant action is composed of the traces of the plaquette variables with the partition sum given by

$$Z = \int \prod_i dU_i e^{\beta \sum_j tr_f P_j} \tag{1}$$

In order to go over to the dual formulation one introduces the group character expansion

$$exp(\beta tr_f P_i) = \sum_{j_i} (2j_i + 1) C_{j_i}(\beta) \chi_{j_i}(P_i) \tag{2}$$

Now either group integrations over products of rotation matrices or graphical rules [2] can be used to carry out the link integrations. In $d = 3$, every link is shared by 4 plaquettes and the link integral is nontrivial if the representations attached to the four plaquettes are such that $j_1 \otimes j_2 \otimes j_3 \otimes j_4$ has a singlet in it. By introducing an auxiliary j_5, this condition can be stated as $(j_1, j_2, j_5), (j_3, j_4, j_5)$ should satisfy the triangle inequalities. These are the analog of Gauss's law constraints of the con-

ventional formulation. For lack of space, I'll just quote the final result [1]

$$Z_d = \sum_{\{j\}} \prod (2j+1) C_{j_a}(\beta) \prod_{i=1}^{5} \begin{Bmatrix} a_i & b_i & c_i \\ d_i & e_i & f_i \end{Bmatrix} \quad (3)$$

This partition sum is defined over the dual lattice where $\{j_a\}$ live over the links and $\{j_b\}$ over the diagonals to the plaquettes. The convention for the diagonals is that they connect the vertices of the odd sublattice.We have collectively designated $\{j_a\}, \{j_b\}$ by $\{j\}$. Each cube of the dual lattice is seen to be spanned by 5 tetrahedra of which one is spanned entirely by $\{j_b\}$ while four are spanned by three $\{j_a\}$ and three $\{j_b\}$. Each tetrahedron carries a weight factor which is the $SU(2)$ 6-j symol $\begin{Bmatrix} a & b & c \\ d & e & f \end{Bmatrix}$. Periodic b.c for the original lattice is crucial for this construction.

The disordered operators D_i satisfy $\chi_{j_i}(P_i) \xrightarrow{D_i} (-1)^{j_i} \chi_{j_i}(P_i)$. The disordered line is composed of the product of disordered operators for each plaquette pierced by the disordered line.

2.1. Connection With Regge-Ponzano Gravity

The partition function of eqn (3) is the same as the partition function of the $d = 3$ Regge-Ponzano gravity except for the C_{j_a} factors. These factors break the invariance under the Alexander moves [1] thereby generating an additional degree of freedom per generator of the gauge group. $SU(2)$ gauge theory in D dimensions has $3(D-2)$ d.o.f. The number of diffeomorphisms being D, the d.o.f left is $2D-6$. This correctly matches the d.o.f of gravity in $D = 3, 4$ dimensions.

3. Numerical Simulations

Apart from the new feature of simulating in the dual variables, there are many practical advantages also. The basic variables are all integers. Further, since $C_j(\beta) \overset{\beta\to\infty}{\longrightarrow} e^{-\frac{j(j+1)}{2\beta}}$, only $j_{max} \simeq \sqrt{\beta}$ is needed making short integers to suffice.

3.1. Updating

An allowed configuration is where various triangle inequalities are satisfied and any updating of variables has to maintain this. Updating $\{j\}$ independently and rejecting configurations that are not allowed is wasteful. First we describe what we call a **local** updating algorithm. Each a-link is common to 4 triangles and each b-link is common to 6 triangles.To update a link, one first forms $j^i_{max} = j^i_1 + j^i_2$ for each triangle i containing the link and where $j_{1,2}$ are the reps on the other two links of the triangle. Likewise $j^i_{min} = |j^i_1 - j^i_2|$ are also calculated and subsequently $j_{max} = min\{j^i_{max}\}$ and $j_{min} = max\{j^i_{min}\}$. Now the allowed values for the link to be updated are in the range (j_{min}, j_{max}) in steps of unity. A particular value can be chosen by either the heatbath or the Metropolis algorithms.

A major problem with this updating is that it does not change a link from integer values(bosonic) to half-integer values(fermionic) and vice versa. Therefore it is not ergodic. If we think of the integer and half-integer values as a Z_2-grading, then these Z_2 d.o.f are not updated by the local method.

Changing only one link from fermionic to bosonic or vice versa is not possible. In any given triangle at least two such changes must be made. Continuing this way one is generically led to a proliferation of changes! We must seek the smallest volume to which such changes can be restricted. If the Z_2 flippings are carried out at an odd site, the minimum volume is 8 cubes. These flippings called **Quasi-local updates** amount to keeping all the links on the outer surface fixed and flipping all the interrior links. If on the other hand the flippings are done at an even site, the minimum volume is a cluster of 8 tetrahedra.

One still faces the question of how these flippings are to be done satisfying all the triangle inequalities. That brings us to the use of Kagome(K) variables [3]. The basic idea is that the triangle inequalities can be traded for *equalities*. If (j_1, j_2, j_3) are the links of a triangle, introducing the variables $j_1 = n_2 + n_3, j_2 = n_3 + n_1, j_3 = n_1 + n_2$, it can be shown that choosing n_i freely among positive integers, one can generate all the allowed j_i. But for our case, if a link

is shared between two triangles, then the sum of K-variables for this link from one triangle must equal the sum of the corresponding K-variables for the other triangle. Thus K-variables are not all *independent*.

Now the solution to the quasi-local update problem in terms of the K-variables is as follows: i) determine the K-variables for the links on the outer surface and keep them fixed; ii) determine the *the set of independent K-variables* for the interrior; iii) change the interrior variables freely subject to realising the fermion-boson flippings for the interrior links. A *simplified* quasi-local update consists of simply shifting all $\{j\}$ by $\{j \pm 1/2\}$.

The Z_2-fluctuations can be controlled by switching on/off the quasi-local updates at any site one wishes. Thermalisation can be achieved first with the local updates and on the resulting ensemble one can perform the quasi-local updates till the Z_2-configurations thermalise. In this manner the precise influence of the Z_2-fluctuations on various dynamical mechanisms can be studied.

Finally it is worth mentioning as to how best the 6-j evaluations should be done. Calculating them for each configuration is unnecessarily time consuming. For local updates it is possible to use a recursive algorithm. But it is best to make a table of only the non-vanishing 6-j's into an one-dimensional array with an unique labelling.

3.2. Positivity of Weights?

The product of 6-j symbols is generally not of a particular sign. Nevertheless, based on my experiences with the $d = 4$ hypercube problem [2] it was initially hoped that a configuration of positive weight remains so under all updates. Over a large number of sweeps this was indeed so, but eventually negative weight configurations started appearing. Barring a phase error in the analytical calculations, it appeared as if the dual formulation had seen the end of the line at least as far as numerical simulations were concerned.

But a resolution of this problem has recently occurred to me. Let us say the weight factors $p_1, p_2, p_3, ..., p_k, ...$ are such that some of them, say, $p_l, p_m, p_n, ...$ are negative. Now the trick is to generate configurations with the probabilities $\{\tilde{p}_i = \frac{|p_i|}{P}\}$ where $P = \sum_i |p_i|$ but keep track of the configurations with negative weight through ϵ_i which is $+1$ when $p_i > 0$ and -1 when $p_i < 0$.The expectation value of an observable with value O_i in the configuration i is now given by

$$\langle O \rangle_p = \frac{\langle O\epsilon \rangle_{\tilde{p}}}{\langle \epsilon \rangle_{\tilde{p}}} \tag{4}$$

Since the partition sum $\langle 1 \rangle_p = \langle \epsilon \rangle_{\tilde{p}} \neq 0$, this procedure is well defined.

4. Open Issues

Among the important open problems are the generalisation of our algorithms to $d = 4$ using the results of [4]. It would also be interesting to generalise our results for the case of the models considered by Ooguri[5] and their non-topological versions. Inclusion of matter particularly fermions is another outstanding problem. Since we are eventually interested only in the large β regime a simpler version using only the asymptotic forms of the 6-j symbols may be possible. Generalisation to $SU(3)$ is another open direction. Finally mention must be made of what I call *exceptional configurations* for which some of the 6-j's vanish even though all the triangle inequalities are satisfied. There are fascinating group theoretical explanations for these. Their contribution to the partition sum is zero. One may speculate as to the physical significance of these configurations for gauge theories.

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